PERGAMON

Applied Mathematics Letters 14 (2001) 897–901

Applied Mathematics Letters

www.elsevier.nl/locate/aml

The Convergence of Sampling Series Based on Multiresolution Analysis

WENCHANG SUN AND XINGWEI ZHOU
Nankai Institute of Mathematics
Nankai University, Tianjin 300071, P.R. China
<sunwch><xwzhou>@nankai.edu.cn

(Received June 2000; accepted July 2000)

Communicated by P. Butzer

Abstract—The sampling theory says that a function may be determined by its sampled values at some certain points provided the function satisfies some certain conditions. In this paper, we consider the convergence of the sampling series for bounded functions with the sampling function from some multiresolution analysis. We also present a convergence theorem on the sampling series with irregular sampling points and give the approximation error.

Keywords—Sampling series, Sampling theorem, Wavelets, Multiresolution analysis.

1. INTRODUCTION

The sampling theory says that if a function $f(x)$ satisfies some certain conditions, then $f(x)$ is uniquely determined by its sampled values at some discrete points $\{x_k : k \in \mathbf{Z}\}$ and

$$f(x) = \sum_{k\in\mathbf{Z}} f(x_k)\, S_k(x),$$

for some sampling functions $\{S_k(x)\}$. For example, if $f \in L^2(\mathbf{R})$ with supp $\hat{f} \subset [-\Omega, \Omega]$, then $f(x) = \sum_{k\in\mathbf{Z}} f(k\pi/\Omega)(\sin(\Omega x - k\pi)/(\Omega x - k\pi))$. This is the classical Shannon sampling theorem. Recently, the sampling theorem has been extended to some wavelet subspaces (see [1–8]). Specifically, we have the following proposition.

PROPOSITION 1.1. *(See [4,7].) Let V_0 be a closed subspace of $L^2(\mathbf{R})$ and $\{\varphi(\cdot - n) : n \in \mathbf{Z}\}$ be a Riesz basis for V_0. Suppose that φ is continuous, $\sup_x \sum_{n\in\mathbf{Z}} |\varphi(x+n)|^2 < +\infty$, and that there exist two constants $C_1, C_2 > 0$ such that*

$$C_1 \le |Z\varphi(0,\omega)| \le C_2, \tag{1.1}$$

where $Z\varphi(x,\omega) := \sum_{n\in\mathbf{Z}} \varphi(x+n)e^{-in\omega}$ is the Zak transform of φ. Put $\hat{S}(\omega) = \hat{\varphi}(\omega)/Z\varphi(0,\omega)$. Then $\{S(\cdot - n)\}$ is a Riesz basis for V_0 and for any $f \in V_0$, $f(x) = \sum_{k\in\mathbf{Z}} f(k)S(x-k)$, where the convergence is both in $L^2(\mathbf{R})$ and uniform on $\mathbf{R}$.

The authors would like to thank the referees for their valuable suggestions which helped us to improve this paper. This work was supported by the National Natural Science Foundation of China (Grant No. 16971047), China Postdoctoral Science Foundation, and the Research Fund for the Doctoral Program of Higher Education.

PII: S0893-9659(01)00062-3

For any $k \geq 0$, define

$$(R_k f)(t) = \sum_{n\in\mathbf{Z}} f\left(\frac{n}{2^k}\right) S\left(2^k t - n\right). \tag{1.2}$$

Fischer [9] proved that if f and φ are smooth enough then $(R_k f)(t)$ converges to $f(t)$ uniformly on $\mathbf{R}$. In literature, the series in (1.2) is usually viewed as a sampling series.

In this paper, we study the convergence of $R_k f$ for any bounded function f. We also study the case for which the sampling points are irregular and give the convergence rate for some continuous functions.

NOTATIONS. In this paper, the Fourier transform of $f \in L^2(\mathbf{R})$ is defined by $\hat{f}(\omega) = \int_{\mathbf{R}} f(x) e^{-ix\omega}\, dx$. $AC_{\text{loc}} = \{f(x) : f \text{ is locally absolutely continuous on } \mathbf{R}\}$.

2. MAIN RESULTS

We begin with a simple lemma.

LEMMA 2.1. *Suppose that $f \in AC_{\text{loc}}$ and f, $f' \in L^1(\mathbf{R})$. Then for any bounded real sequence $\{c_n\}$, $\sum_{n\in\mathbf{Z}}|c_n f(x-n)|$ converges pointwisely to a continuous function.*

PROOF. Let $C = \sup_n |c_n|$. Since $\int_0^1 \sum_n |c_n f(x-n)|\, dx \leq \int_0^1 C \cdot \sum_n |f(x-n)|\, dx = C\|f\|_{L^1} < +\infty$, $\sum_{n\in\mathbf{Z}}|c_n f(x-n)|$ is convergent almost everywhere on $\mathbf{R}$. On the other hand, for any $x < y < x+1$, we see from $f' \in L^1(\mathbf{R})$ that $\sum_{n\in\mathbf{Z}}|c_n f'(t-n)|$ is convergent in $L^1[x,y]$ and $|\sum_{n\in\mathbf{Z}}|c_n f(y-n)| - \sum_{n\in\mathbf{Z}}|c_n f(x-n)|| \leq \int_x^y \sum_{n\in\mathbf{Z}}|c_n f'(t-n)|\, dt$. Hence, $\sum_{n\in\mathbf{Z}}|c_n f(x-n)|$ converges pointwisely to a continuous function.

THEOREM 2.2. *Let $\varphi(x)$ be continuous such that*

(i) *$\varphi \in AC_{\text{loc}}$ and φ, $\varphi' \in L^1(\mathbf{R})$, or*
(ii) *there exist some C, $\varepsilon > 0$ such that $|\varphi(x)| \leq C \cdot (1/(1+|x|^{1+\varepsilon}))$.*

Suppose that $Z\varphi(0,\omega)$ satisfies (1.1) for some constants C_1,$C_2 > 0$. Let $\hat{S}(\omega) = \hat{\varphi}(\omega)/Z\varphi(0,\omega)$. If $\hat{\varphi}(2n\pi) = \delta_{n,0}$, then for any bounded function f and each point $t \in \mathbf{R}$ of continuity of f, $\lim_{k\to+\infty}(R_k f)(t) = f(t)$. Furthermore, if f is uniformly continuous on $\mathbf{R}$, then the convergence is uniform on $\mathbf{R}$.

PROOF. First, we show that $\sum_{k\in\mathbf{Z}}|S(x-k)|$ is convergent uniformly on $[0,1]$.

For Condition (i), we see from Lemma 2.1 that $\sum_{n\in\mathbf{Z}}|\varphi(n)| < +\infty$. For Condition (ii), it is easy to see that $\sum_{n\in\mathbf{Z}}|\varphi(n)| < +\infty$. Since $Z\varphi(0,\omega)$ has no zero on $\mathbf{R}$, by Wiener's lemma [10, pp. 367,368], there exists some $\{\alpha_k\} \in l^1$ such that $1Z\varphi(0,\omega) = \sum_{n\in\mathbf{Z}} \alpha_n e^{-in\omega}$. Hence, $S(x) = \sum_{n\in\mathbf{Z}} \alpha_n \varphi(x-n)$.

Let $S_1(x) = \sum_{n\in\mathbf{Z}}|\alpha_n \varphi(x-n)|$. For Condition (i), $S_1(x)$ is continuous due to Lemma 2.1 and

$$\sum_{k\in\mathbf{Z}} S_1(x-k) = \sum_{k\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} |\alpha_n\varphi(x-k-n)| = \sum_{n\in\mathbf{Z}}|\alpha_n|\sum_{k\in\mathbf{Z}}|\varphi(x-k)|.$$

Using Lemma 2.1 again, we see that $\sum_{k\in\mathbf{Z}} S_1(x-k)$ is continuous on R. For Condition (ii), since $\sum_{n\in\mathbf{Z}}|\varphi(x-n)|$ is convergent uniformly on any compact set, it is easy to see that both $S_1(x)$ and $\sum_{k\in\mathbf{Z}}S_1(x-k)$ are continuous on $\mathbf{R}$.

By Dini's theorem [11, pp. 369,370], $\sum_{k\in\mathbf{Z}}S_1(x-k)$ is convergent uniformly on $[0,1]$. Since $|S(x)| \leq S_1(x)$, $\sum_{k\in\mathbf{Z}}S(x-k)$ is also convergent uniformly on $[0,1]$.

Next, we show that $\sum_{k\in\mathbf{Z}} S(x-k) = 1$, $\forall x$.

For any $k \in \mathbf{Z}$, $\int_0^1 \sum_{n\in\mathbf{Z}} \varphi(x-n)e^{-i2k\pi x}\, dx = \int_{-\infty}^{+\infty} \varphi(x)e^{-i2k\pi x}\, dx = \hat{\varphi}(2k\pi) = \delta_{k,0}$. Hence, $\sum_{n\in\mathbf{Z}} \varphi(x-n) \equiv 1$, $\forall x$. In particular, $Z\varphi(0,0) = \sum_{n\in\mathbf{Z}}\varphi(n) = 1$ and so $\sum_{k\in\mathbf{Z}}\alpha_k = 1/(Z\varphi(0,0)) = 1$. It follows that $\sum_{k\in\mathbf{Z}} S(x-k) = \sum_{k\in\mathbf{Z}}\sum_{n\in\mathbf{Z}} \alpha_n\varphi(x-k-n) = \sum_{n\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \alpha_n\varphi(x-k-n) = 1$.

Now the conclusion follows by Theorem 1 in [12].

Next we study the convergence of sampling series for which the sampling points are irregular. Let $N\ell^1$ be the set of bounded functions $f(x)$ such that $\sum_{n\in\mathbf{Z}}|f(x-n)|$ is convergent uniformly on $[0,1]$ and that $\sum_{n\in\mathbf{Z}}f(x-n)\equiv 1$.

LEMMA 2.3. *Suppose that $h>0$ and that $\{a(h,n): n\in\mathbf{Z}\}$ is a real sequence satisfying*

$$\lim_{h\to 0}\sup_n |a(h,n)-hn| = 0. \tag{2.1}$$

Then for any $S\in N\ell^1$ and $f\in L^\infty$, at each point $t\in\mathbf{R}$ of continuity of f,

$$\lim_{h\to 0}\sum_{n\in\mathbf{Z}} f(a(h,n))S\left(\frac{t}{h}-n\right) = f(t).$$

PROOF. Let $M=\max\{\|f\|_\infty, \sup_x\sum_{n\in\mathbf{Z}}|S(x-n)|\}$. Then $M<+\infty$.

For any $\varepsilon>0$, there exists some $N>0$ such that

$$\sum_{|x/h-n|\ge N}\left|S\left(\frac{x}{h}-n\right)\right|<\varepsilon, \qquad \forall x,h.$$

Since f is continuous at t, there exists some $\delta>0$ such that for any x with $|x-t|<\delta$, $|f(x)-f(t)|<\varepsilon$. On the other hand, by (2.1), there exists some $0<\eta<\delta/2N$ such that for any $0<h<\eta$, $|a(h,n)-hn|<\delta/2, \forall n$. It follows that $|a(h,n)-t|\le|a(h,n)-hn|+|hn-t|<\delta/2+Nh\le\delta$ if $|t/h-n|\le N$. Hence,

$$\begin{aligned}
&\left|\sum_{n\in\mathbf{Z}} f(a(h,n))S\left(\frac{t}{h}-n\right)-f(t)\right| \\
&\quad\le \sum_{|t/h-n|<N}|f(a(h,n))-f(t)|\cdot\left|S\left(\frac{t}{h}-n\right)\right| + \sum_{|t/h-n|\ge N}|f(a(h,n))-f(t)|\cdot\left|S\left(\frac{t}{h}-n\right)\right| \\
&\quad\le \varepsilon\cdot M+2M\varepsilon = 3M\varepsilon, \qquad 0<h<\eta.
\end{aligned}$$

This completes the proof.

By Theorem 2.2 and Lemma 2.3, it is easy to prove the following theorem.

THEOREM 2.4. *Let the hypotheses be as in Theorem 2.2. Furthermore, suppose that $\{x_n\}$ is a real sequence such that $|x_n-n|\le L$ for some constant L. Then for any bounded function $f(t)$, at each point $t\in\mathbf{R}$ of continuity of f, $\lim_{k\to+\infty}\sum_n f(x_n/2^k)S(2^kt-n)=f(t)$.*

At the end of this paper, let us study the approximation error. The following two lemmas are easy to prove, and we leave them to the readers.

LEMMA 2.5.

$$\sum_{n\in\mathbf{Z}}\frac{1}{(1+|n-x|)^\alpha}\le B(\alpha):=1+\frac{\alpha+2}{\alpha-1}\left(\frac{2}{3}\right)^\alpha, \qquad \forall\alpha>1, \quad x\in\mathbf{R}.$$

LEMMA 2.6. *Suppose that $\varphi\in L^2(\mathbf{R})$ and $\hat\psi(\omega)=\beta(\omega)\hat\varphi(\omega)$, where $\beta(\omega)$ is a bounded 2π-periodic function. Then $\sum_{n\in\mathbf{Z}}|\psi(x-n)|^2\le\|\beta\|_\infty^2\sum_{n\in\mathbf{Z}}|\varphi(x-n)|^2$, a.e.*

THEOREM 2.7. *Let the hypotheses be as in Theorem 2.4. Define $B(\cdot)$ as in Lemma 2.5. If $|\varphi(x)|\le C(1+|x|)^{-\gamma}$, $\gamma>2$, and $|f(x)-f(y)|\le C_f|x-y|^\alpha, 0<\alpha<1/2$, then*

$$\left|f(t)-\sum_{n\in\mathbf{Z}}f\left(\frac{x_n}{2^k}\right)S\left(2^kt-n\right)\right|\le 2^{-\alpha k}C_f\left(L^\alpha M_{0,\gamma}+M_{\alpha,\gamma}\right), \qquad \forall t\in\mathbf{R},$$

where

$$M_{\alpha,\gamma} = \frac{2CB(\gamma)}{C_1} + \sqrt{\frac{4(1-\alpha)}{1-2\alpha}}\left[\frac{2C^2\sqrt{B(2\gamma)}}{C_1^2(\gamma-2)} + \frac{C\sqrt{B(2\gamma-2)}}{C_1}\right].$$

PROOF. Put $\beta(\omega) = 1/Z\varphi(0,\omega) = \sum_{k\in\mathbf{Z}} \beta_k e^{-ik\omega}$. Then

$$\|\beta'(\omega)\|_\infty = \left\|\frac{(Z\varphi(0,\omega))'}{(Z\varphi(0,\omega))^2}\right\|_\infty \le \frac{1}{C_1^2}\sum_{k\in\mathbf{Z}} |k\varphi(k)| \le \frac{C}{C_1^2}\sum_{k\neq 0}\frac{1}{(1+|k|)^{\gamma-1}} \le \frac{2C}{C_1^2(\gamma-2)}$$

and $\|\beta(\omega)\|_\infty \le 1/C_1$. It follows that

$$|S(x)| = \left|\sum_{k\in\mathbf{Z}} \beta_k\varphi(x-k)\right| \le \|\beta(\omega)\|_\infty \sum_{k\in\mathbf{Z}} \frac{C}{(1+|x-k|)^\gamma} \le \frac{CB(\gamma)}{C_1}.$$

Since $[-ixS(x)]\hat{}(\omega) = \hat{S}'(\omega) = \beta'(\omega)\hat{\varphi}(\omega) + \beta(\omega)\hat{\varphi}'(\omega)$, by Lemma 2.6, we have

$$\begin{aligned}
&\left(\sum_{n\in\mathbf{Z}} |(x-n)S(x-n)|^2\right)^{1/2} \\
&\le \|\beta'\|_\infty \left(\sum_{n\in\mathbf{Z}} |\varphi(x-n)|^2\right)^{1/2} + \|\beta\|_\infty \left(\sum_{n\in\mathbf{Z}} |(x-n)\varphi(x-n)|^2\right)^{1/2} \\
&\le \|\beta'\|_\infty C \left(\sum_{n\in\mathbf{Z}} (1+|x-n|)^{-2\gamma}\right)^{1/2} + \|\beta\|_\infty C \left(\sum_{n\in\mathbf{Z}} (1+|x-n|)^{-2\gamma+2}\right)^{1/2} \\
&\le \frac{2C^2\sqrt{B(2\gamma)}}{C_1^2(\gamma-2)} + \frac{C\sqrt{B(2\gamma-2)}}{C_1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n\in\mathbf{Z}} |x-n|^\alpha |S(x-n)| &\le 2\|S\|_\infty + \sum_{|n-x|\ge 1} |x-n|^{\alpha-1}|(x-n)S(x-n)| \\
&\le 2\|S\|_\infty + \left(\sum_{|n-x|\ge 1} |x-n|^{2\alpha-2}\right)^{1/2} \left(\sum_{n\in\mathbf{Z}} |(x-n)S(x-n)|^2\right)^{1/2} \\
&\le \frac{2CB(\gamma)}{C_1} + \sqrt{\frac{4(1-\alpha)}{1-2\alpha}}\left[\frac{2C^2\sqrt{B(2\gamma)}}{C_1^2(\gamma-2)} + \frac{C\sqrt{B(2\gamma-2)}}{C_1}\right] = M_{\alpha,\gamma}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|f(t) - \sum_{n\in\mathbf{Z}} f\left(\frac{x_n}{2^k}\right) S\left(2^k t - n\right)\right| &= \left|\sum_{n\in\mathbf{Z}} \left[f\left(\frac{x_n}{2^k}\right) - f(t)\right] S\left(2^k t - n\right)\right| \\
&\le \sum_{n\in\mathbf{Z}} C_f \left|x_n - 2^k t\right|^\alpha 2^{-\alpha k} \left|S\left(2^k t - n\right)\right| \\
&\le \sum_{n\in\mathbf{Z}} 2^{-\alpha k} C_f \left(|x_n - n|^\alpha + |2^k t - n|^\alpha\right) \left|S\left(2^k t - n\right)\right| \\
&\le 2^{-\alpha k} C_f L^\alpha \sum_{n\in\mathbf{Z}} \left|S\left(2^k t - n\right)\right| \\
&\quad + 2^{-\alpha k} C_f \sum_{n\in\mathbf{Z}} \left|2^k t - n\right|^\alpha \left|S\left(2^k t - n\right)\right| \\
&\le 2^{-\alpha k} C_f \left(L^\alpha M_{0,\gamma} + M_{\alpha,\gamma}\right).
\end{aligned}$$

REMARK. The approximation order can also be derived from results in [12] or [13]. Here we give the explicit error bound. We point out that both scaling functions $\varphi_N (2 \le N \le 20)$ of Daubechies' and B-splines defined by

$$\hat{\varphi}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2} \right)^{m+1}, \qquad m \ge 1,$$

meet the requirements of these theorems (see [8]).

REFERENCES

1. A. Aldroubi and M. Unser, Families of wavelet transforms in connection with Shannon's sampling theory and the Gabor transform, In *Wavelets—A Tutorial in Theory and Applications*, (Edited by C.K. Chui), pp. 509–528, Academic Press, Boston, MA, (1992).
2. A.J.E.M. Janssen, The Zak transform and sampling theorem for wavelet subspaces, *IEEE Trans. Signal Proc.* **41**, 3360–3364 (1993).
3. Y. Liu and G. Walter, Irregular sampling in wavelet subspaces, *Fourier Anal. Appl.* **2**, 181 189 (1995).
4. W. Sun and X. Zhou, Frames and sampling theorem, *Science in China* **41**, 606–612 (1998).
5. W. Sun and X. Zhou, Sampling theorem for multiwavelet subspaces, *Chinese Science Bulletin* **44**, 1283–1286 (1999).
6. X.G. Xia and Z. Zhang, On sampling theorem, wavelets, and wavelet transforms, *IEEE Trans. Signal Proc.* **41**, 3524–2535 (1993).
7. G. Walter, A sampling theorem for wavelet subspaces, *IEEE Trans. Inform. Theory* **38**, 881–884 (1992).
8. X. Zhou and W. Sun, On the sampling theorem for wavelet subspaces, *J. Fourier Anal. Appl.* **5**, 347–354 (1999).
9. A. Fischer, Multiresolution analysis and multivariate approximation of smooth signals in $C_B(\mathbf{R}^d)$, *J. Fourier Anal. Appl.* **2**, 161–180 (1995).
10. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, (1987).
11. S.C. Malik, *Mathematical Analysis*, John Wiley & Sons, New York, (1984).
12. P.L. Butzer, S. Ries and R.L. Stens, Approximation of continuous and discontinuous functions by generalized sampling series, *J. Approx. Theory* **50**, 25–39 (1987).
13. A. Fischer, On wavelets and prewavelets with vanishing moments in higher dimensions, *J. Approx. Theory* **90**, 46–74 (1997).